Resonance Scattering and Spatial Variation of the Order Parameter in Superconducting Alloys*

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The problem of spatial variation of the order parameter in the neighborhood of a nonmagnetic impurity with broadened d levels is investigated near the superconducting critical temperature. It is shown that, due to resonance scattering, a long-range-order variation (compared to the coherence length) actually occurs and several expressions, obtained through different methods of calculation, are given and analyzed. This spatial variation leads to a small increase of the critical temperature as compared to its magnitude calculated using the average value of the order parameter; the analytical expression for this increase is given explicitly. The physical implications of the theoretical results are discussed on the basis of Anderson's theorem, and the main conclusion is that the incoherent d-state admixture at the Fermi level yields a much more important contribution to the change in thermodynamic properties, because of the presence of impurities, than the one due to the spatial variation of the order parameter.

I. INTRODUCTION

Recently, the spatial variation of the order parameter of a superconductor in the vicinity of an impurity has attracted a great deal of attention. Tsusuki and Tsuneto[1] (TT) first treated the problem for the case of impurities which sustain a magnetic moment; their method of solution presented, however, some disadvantages which were pointed out by Heinrichs[2] (hereafter refered to as H) who at the same time proposed an alternative treatment free of the criticized shortcomings. More recently Kitamura[3] solved the problem of the spatial variation of the order parameter in the neighborhood of a magnetic impurity taking into account the Kondo effect; he used for this purpose a method similar to that of H.[2]

On the other hand, extensive work has been done on superconducting alloys of simple metals with transition elements[4-6] which do not sustain a magnetic moment in the host matrix. (i.e., *Al*Cr, *Al*Mn). Experimentally[7] a marked decrease in the critical temperature is observed which has been interpreted on the basis of resonance scattering; the analytical treatment is carried out using a model due to Anderson.[8] This treatment is also successful in explaining other physical properties like residual resistivity, static susceptibility, thermopower, and specific heat.

At the same time, according to Anderson's theorem[9] no marked variation of the superconducting critical temperature should be observed unless (a) time-reversal invariance of the Hamiltonian is destroyed, or (b) there is a long-range spatial variation of the order parameter. But, when the proof of this theorem is examined, one notices that it is taken for granted that the density of states at the Fermi level remains nearly constant in the presence of the external perturbation, provided time-reversal invariance is preserved.[10] However, this is certainly not the case when one deals with resonance scattering since the broadened atomic d states of the transition element contribute significantly to the density of states at the Fermi level. Furthermore, because no phonon coupling is assumed to occur between d electrons, in the superconducting state of the alloy the d-state admixture at the Fermi surface is incoherent. In this paper we show that this incoherent admixture is a much more important cause of change for the thermodynamic properties of our superconducting alloy, as compared with the pure-metal case, than the spatial variation of the order parameter. Both contributions to the change in critical temperature are explicitly evaluated in Sec. III, and their relative orders of magnitude are estimated. At the same time it is shown that there is a non-negligible long-range spatial variation of the order parameter and the corresponding analytic form is given explicitly.

From the formal point of view, we start from the Abrikosov and Gorkov[11] integral equation for the order parameter $\Delta(\vec{r})$, valid near the transition temperature T_c of the alloy and use the Hartree-Fock solution to the Anderson[8] Hamiltonian to for-

mulate the problem analytically. The precise details are given in Sec. II below. In Sec. III a perturbation-theory treatment is carried out following the scheme of H; at the same time the decrease in critical temperature, due to the presence of impurities, is obtained on the basis of an average value of the order parameter, which leads to the recovery of well-known relations.[6] The correction, due to spatial variation of the order parameter, is obtained afterwards. In Sec. IV a different, but simpler, approach similar to the one proposed by TT is used in order to find the spatial dependence of the order parameter at large distance from the impurity. In Sec. V conclusions are drawn and extensions of this work are suggested.

II. MATHEMATICAL FORMULATION

The integral equation for the order parameter $\Delta(\vec{r})$, in the presence of a single nonmagnetic impurity with broadened d levels, is similar in form to the one that holds in the case of magnetic impurities.[1] The linearized relation, valid close to the critical temperature T_c, was first derived by Abrikosov and Gorkov[11] and reads[12]

$$\Delta(\vec{r}) = gT\sum_n \int d^3l\, G(\vec{r},\vec{l},\omega_n)\Delta(\vec{l})G(\vec{l},\vec{r},-\omega_n), \quad (1)$$

where g is the positive BCS[13] coupling constant, $G(\vec{r},\vec{l},\omega_n)$ is the thermodynamic Green's function for an electron in the normal metal in the presence of the impurity, and $\omega_n = \pi T(2n+1)$, with n being an integer.

Let us first consider the case of a single scattering center; we locate it at the origin of our coordinate system. The model we use to represent this physical system is due to Anderson,[8] and we simply take the Hartree-Fock solution for the full propagator. Thus, in the momentum representation one can write (see also Fig. 1)

$$G(\vec{k},\vec{k}',\omega_n) = \frac{\delta_{\vec{k},\vec{k}'}}{i\omega_n - \epsilon_k} + \frac{1}{i\omega_n-\epsilon_k} \times V_{kd}\frac{1}{i(\omega_n+\Gamma_n)-E_d}V_{dk'}\frac{1}{i\omega_n-\epsilon_{k'}}, \quad (2)$$

where $\delta_{\vec{k}\vec{k}'}$ is the Kronecker symbol, ϵ_k the s-electron kinetic energy measured, as all energies throughout this paper, from the Fermi level ϵ_F, and V_{kd} is the s-d overlap integral. $\Gamma_n = \Gamma\,\mathrm{sign}(n)$, with the width Γ of the d states, broadened by the s-d interaction, given by

$$\Gamma = \pi N(0)\langle|V_{kd}|^2\rangle. \quad (3)$$

The average is taken over the Fermi surface; $N(0) = mp_F/2\pi^2$ is the one-spin s-electron density of states at the Fermi surface, where p_F is the Fermi momentum, and m is the electron mass.

The propagator G can easily be transformed into configuration space to yield

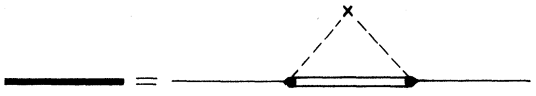

FIG. 1. Graphic representation of Eqs. (2), (4), and (6). The solid line stands for the full s-electron propagator in the Hartree-Fock approximation G, while the simple line represents the free-electron propagator G_0. The × stands for the impurity and the double line for the "resonant d-electron" Green's function in the Hartree-Fock approximation G_d.

$$G(\vec{r},\vec{r}',\omega_n) = G_0(|\vec{r}-\vec{r}'|,\omega_n) + \int \frac{d^3k}{(2\pi)^3}\frac{V_{kd}e^{i\vec{k}\cdot\vec{r}}}{i\omega_n-\epsilon_k}\frac{1}{i(\omega_n+\Gamma_n)-E_d} \times \int \frac{d^3k'}{(2\pi)^3}\frac{V_{dk'}e^{-i\vec{k}'\cdot\vec{r}'}}{i\omega_n-\epsilon_{k'}}, \quad (4)$$

where

$$G_0(R,\omega_n) = -\frac{m}{2\pi R}\exp\left(ip_F\,\mathrm{sign}(n) - \frac{|\omega_n|}{v_F}\right)R \quad (5)$$

is the bare free-electron propagator in configuration space, with p_F and v_F standing for the Fermi momentum and velocity, respectively.

The matrix element for the s-d interaction is assumed by Anderson[8] not to change radically when $\vec{k}$ varies in direction; the same approximation is made here. We obtain, in this way, the following expression for the full electron propagator in configuration space:

$$G(\vec{r},\vec{r}',\omega_n) \cong G_0(|\vec{r}-\vec{r}'|,\omega_n) + \frac{\Gamma}{\pi N(0)}G_0(r,\omega_n)G_d(\omega_n)G_0(r',\omega_n), \quad (6)$$

where $G_d(\omega_n)$ is the "resonant d-electron" propagator in the self-consistent field (Hartree-Fock) approximation. Its analytical form is

$$G_d(\omega_n) = [i(\omega_n+\Gamma_n)-E_d]^{-1}, \quad (7)$$

where E_d is the renormalized energy of the electron virtually bound to the impurity.

As in H and TT we now introduce the following convenient definitions:

$$K_0(|\vec{r}-\vec{l}|,\omega_n) = G_0(|\vec{r}-\vec{l}|,\omega_n)G_0(|\vec{r}-\vec{l}|,-\omega_n),$$

$$I_1(r,\omega_n) = G_0(r,\omega_n)G_0(r,-\omega_n) \times \int d^3l\, G_0(l,\omega_n)G_0(l,-\omega_n)\Delta(l), \quad (8)$$

$$I_2(r,\omega_n) = G_0(r,\omega_n) \times \int d^3l\, G_0(l,\omega_n)G_0(|\vec{r}-\vec{l}|,-\omega_n)\Delta(l).$$

Using all the above introduced quantities, we re-

write the equation for $\Delta(r)$, which now takes the form

$$\Delta(r) - gT\sum_n \int d^3l\, K_0(|\vec{r}-\vec{l}|, \omega_n)\Delta(l)$$
$$= gT\frac{\Gamma}{\pi N(0)}\sum_n \Big(G_d(\omega_n) I_2(r, \omega_n) + G_d(-\omega_n) I_2(r, -\omega_n)$$
$$+ \frac{\Gamma}{\pi N(0)} G_d(\omega_n) G_d(-\omega_n) I_1(r, \omega_n) \Big) . \quad (9)$$

The following relations hold true:

$$I_1(r, \omega_n) = I_1^*(r, \omega_n) = I_1(r, -\omega_n) ,$$
$$I_2(r, \omega_n) = -I_2^*(r, \omega_n) = -I_2(r, -\omega_n) .$$

Furthermore, a remarkable simplification in the calculations can be achieved if the following formula, valid to order $|\omega_n|/\epsilon_F$ and first derived by TT, is used:

$$I_2(r, \omega_n) = \frac{1}{2\pi i N(0)\,\mathrm{sign}(n)} I_1(r, \omega_n) .$$

Now Eq. (9) can be rewritten, in quite a compact form, which reads

$$\Delta(r) - g\,T\sum_n \int d^3l\, K_0(|\vec{r}-\vec{l}|, \omega_n)\Delta(l)$$
$$= -g\,T\sum_n A(\omega_n) K_0(r, \omega_n) \int d^3l\, K_0(l, \omega_n)\Delta(l) , \quad (10)$$

where we have introduced the definition

$$A(\omega_n) = \frac{1}{[\pi N(0)]^2} \frac{|\omega_n|}{(\omega_n + \Gamma_n)^2 + E_d^2} . \quad (11)$$

It should be emphasized that the main part of the terms on the right-hand side of Eq. (9) corresponds to time-reversed contributions which later on cancel each other out, as expected, since our Hamiltonian is time-reversal invariant. This fact is of vital importance for the perturbation-theory results which are obtained later.

As suggested in H we rewrite (10) in operator form

$$L(r)\Delta(r) = -\frac{TV}{N(0)}\sum_n A(\omega_n) K_0(r, \omega_n) \langle K_0(l, \omega_n)\Delta(l)\rangle , \quad (12)$$

with the notations

$$L(r)\Delta(r) = \frac{1}{g N(0)} \times \Big(\Delta(r) - g\,T\sum_n \int d^3l\, K_0(|\vec{r}-\vec{l}|, \omega_n)\Delta(l)\Big)$$

and

$$\langle f\rangle = (1/V)\int d^3r f(\vec{r}) ,$$

where V is the volume of the system.

III. PERTURBATION-THEORY RESULTS

A. General Case

Following the method of Heinrichs, we seek a perturbation-theory solution to Eq. (11). For this purpose we define a constant parameter z which satisfies

$$z\langle\Delta(r)\rangle = \frac{TV}{N(0)}\sum_n A(\omega_n)\langle K_0(r, \omega_n)\rangle\langle K_0(l, \omega_n)\Delta(l)\rangle \quad (13)$$

and a straightforward calculation gives us an equation to determine the change in transition temperature, due to the impurity, which reads

$$z = \ln(T_{c0}/T_c) , \quad (13')$$

where T_{c0} and T_c are the critical temperatures of the pure metal and the alloy, respectively.

It is convenient to define a function $f(r)$ which represents the deviation of the order parameter from its constant value, due to the presence of the impurity. Analytically,

$$\Delta(r) = [1 + f(r)]\Delta_0 ,$$

with $\Delta_0 \equiv \Delta(\infty)$.

Now we start the perturbation-theory procedure proper by expanding both $f(r)$ and z in series,

$$f(r) = f_1(r) + f_2(r) + \cdots ,$$
$$z = z_1 + z_2 + \cdots ,$$

where the pairs $f_n(r)$, z_n are of order A^n. In first order one readily obtains the dominant term $f_1(r)$:

$$f_1(r) \cong -\frac{TV}{N(0)}\sum_n A(\omega_n)[L(r) + z]^{-1} \times K_0(r, \omega_n)\langle K_0(l, \omega_n)\rangle . \quad (14)$$

This equation serves as a starting point for a long calculation, which is quite similar to the one outlined in H; it yields the following approximate expression for the order parameter:

$$f_1(\rho) \cong -\frac{4\gamma}{(p_F\xi)^2}\frac{1}{\rho}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(2l+1+\gamma)^2 + d^2}\frac{1}{l_0^2(2ni) + \frac{1}{4}\pi^2}\left(\ln\left|\frac{2n+(2l+1)}{2n-(2l+1)}\right|\right)e^{-2n\rho}$$
$$-\frac{2\gamma}{(p_F\xi)^2}\frac{1}{\rho^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1+\gamma)^2+d^2}\left(\frac{4}{\pi^2}\{[\ln(2l+1)]\,e^{-(2l+1)\rho} + E_1[(2l+1)\rho] - E_1(4.8\rho)\} - \frac{\cos(2p_F\xi\rho)}{\ln(2p_F\xi)}\,e^{-(2l+1)\rho}\right). \quad (15)$$

where $\rho = r/\xi$ is the distance from the impurity in units of $\xi = v_F/2\pi T$; ξ is approximately equal to (i.e., only slightly larger) the superconducting coherence length $\xi_0 = v_F/2\pi T_{c0}$. The parameters γ and d are defined as

$$\gamma = \Gamma/\pi T, \quad d = E_d/\pi T ,$$

and the function $l_0(t)$ is given by

$$l_0(t) = \frac{1}{it} \ln\left|\frac{\Gamma(\frac{1}{2}+\frac{1}{2}it)}{\Gamma(\frac{1}{2}-\frac{1}{2}it)}\right| - \psi(\tfrac{1}{2}) ,$$

where $\Gamma(x)$ and $\psi(x)$ are the γ and di-γ function,[14] respectively, and $\gamma(\frac{1}{2}) = -\gamma_E - 2\ln 2$ with $\gamma_E = 0.57721\ldots$ standing for the Euler-Mascheroni constant. Finally, $E_1(x) = \int_x^\infty e^{-t}t^{-1}\,dt$ is the exponential integral.[14]

It should be noted that in our work there is no need to introduce cutoffs anywhere. This is very satisfying, since when one calculates only relative changes in physical properties of superconductors, the divergences that appear in the relations valid separately (for example, in the expressions for the critical temperature for the pure metal and the alloy) should cancel each other out.

It is quite clear that Eq. (15) is indeed cumbersome to use directly. Therefore, it is most convenient to study the long- and short-distance asymptotic forms, i.e., $\rho \gg 1$ and $\rho \gg 1$, since the physically relevant length in our problem is the coherence distance ξ_0. We obtain at large distances ($r \gg \xi_0$)

$$f_1(\rho) \cong \frac{1}{(p_F\xi)^2}\frac{\gamma}{\gamma^2+d^2}\left(-2.72\ln\gamma\frac{e^{-2\rho}}{\rho} - \frac{8}{\pi^2}\frac{e^{-\rho}}{\rho^3} + 2\frac{\cos 2p_F\xi\rho}{\ln 2p_F\xi}\frac{e^{-\rho}}{\rho}\right), \quad (16)$$

which for reasons given below is not very satisfactory.

In the opposite limit ($r \ll \xi_0$)

$$f_1(\rho) \cong \frac{2}{(p_F\xi)^2}\frac{1}{\rho}\left[\sum_{n=1}^{\infty}\frac{1}{\ln^2 n + \frac{1}{4}\pi^2}\ln\left(\frac{2n+1}{2n-1}\right) + 2\gamma\sum_{n=1}^{\infty}\frac{1}{(2n+1+\gamma)^2+d^2}\frac{\ln(4n+1)}{\ln^2 n+\frac{1}{4}\pi^2}\right] - \frac{1}{(p_F\xi)^2}\frac{1}{\rho^2}\left(6.40 - \frac{\cos 2p_F\xi\rho}{\ln 2p_F\xi}\right) . \quad (17)$$

Similarly, the first term in the perturbation series of z is given by

$$z_1 = \frac{TV}{N(0)}\sum_n A(\omega_n)\langle K_0(\omega_n)\rangle\langle K_0(\omega_n)\rangle , \quad (18)$$

and it can be shown that in the limit of dilute concentration of impurities, if f is small, the relation between z and the superconducting transition temperatures (13′) takes the following form:

$$\ln(T_{c0}/T_c) = n_I z , \quad (19)$$

where n_I is the atomic concentration of impurities in the host matrix.

The spatial average of the kernel $K_0(r, \omega_n)$ is easily evaluated and is equal to

$$\langle K_0(\omega_n)\rangle = \frac{\pi N(0)}{V}\frac{1}{|\omega_n|} , \quad (20)$$

which substituted in the Eq. (18) for z, gives after combination with (11),

$$\ln\left(\frac{T_{c0}}{T_c}\right) \cong \frac{2n_I T\Gamma}{N(0)}\sum_{n=0}^{\infty}\frac{1}{\omega_n}\frac{1}{(\omega_n+\Gamma)^2+E_d^2} . \quad (21)$$

In the limit $E_d = 0$ the summations above can be carried out exactly and yield the well-known result[15]

$$\ln\left(\frac{T_{c0}}{T_c}\right) \cong n_I\frac{N_d(0)}{N(0)}\left(\psi(\lambda_c) + \gamma_E - \frac{\Gamma}{2\pi T_c}\zeta(2, \lambda_c)\right), \quad (22)$$

where γ_E is the Euler-Mascheroni constant and $N_d(0) \cong 1/\pi\Gamma$ is the d-electron density of states at the Fermi level, $\lambda_c = \frac{1}{2} + \Gamma/2\pi T_c$, $\psi(x)$ is again the di-γ function and $\zeta(s, x)$ is the generalized Riemann ζ function[14] defined as follows:

$$\zeta(s, x) = \sum_{n=0}^{\infty}(n+x)^{-s} .$$

The fact that we have been able to recover a well-established result for the decrease in critical temperature due to resonance scattering using a completely different method, provides a convenient check for the model and approximations used in Sec. II.

Furthermore, it is clearly seen that z_1 yields only the contribution due to the incoherent d-state admixture at the Fermi level. Since the perturbation series for z is rapidly convergent, as shown below, the term z_1 represents the dominant physical effect, which in the case of resonance scattering turns out to be a marked change in density of states at the Fermi surface.

We now set out to calculate the higher-order corrections, due to spatial variation of the order parameter; the series for z can be rewritten as

$$z = z_1(1 + z_2/z_1 + \cdots) ,$$

and therefore all that we are interested in evaluating is z_2/z_1. Using a procedure similar to that of H we obtain after a long calculation

$$\frac{z_2}{z_1} \cong \frac{\Gamma^3 T_c^2}{\pi^2 N^3(0)} \left(\psi(\lambda_c) + \gamma_E - \frac{\Gamma}{2\pi T_c}\,\zeta(2,\lambda_c)\right)^{-1} \left(\sum_{n=0}^{\infty} \frac{I_n}{[(\omega_n+\Gamma)^2+E_d^2]^2} + \sum_{n,n'=0,n\neq n'}^{\infty} \frac{I_{nn'}}{[(\omega_n+\Gamma)^2+E_d^2][(\omega_{n'}+\Gamma)^2+E_d^2]}\right), \quad (23)$$

where

$$I_{nn'} \cong \frac{m^4}{8\pi^3\xi}\left[\sum_{j=1}^{\infty}[l_0^2(2ij)+\tfrac{1}{4}\pi^2]^{-1}\ln\left|\frac{2j+(2n+1)}{2j-(2n+1)}\right|\ln\left|\frac{2j+(2n'+1)}{2j-(2n'+1)}\right| + \left(\pi\,\mathrm{Re}\,E_1(-\ln 2\xi p_F+\tfrac{1}{2}\pi i) - \pi\,\mathrm{Re}\,[-\ln(2n+1)+\tfrac{1}{2}\pi i]\right.\right.$$

$$\left.\left. + (2n'+1)\{\ln[(\ln 2\xi p_F)^2+\tfrac{1}{4}\pi^2] - \ln[\,[\ln(2n+1)]^2+\tfrac{1}{4}\pi^2]\} + \frac{2n'+1}{\ln 2\xi p_F}\ln\frac{2\epsilon_F}{\pi T(2n+1)}\right) + \left(n \rightleftharpoons n'\right)\right],$$

with $n \neq n'$ and for $n = n'$,

$$I_n \cong \frac{m^4}{8\pi^3\xi}\left\{\sum_{j=1}^{\infty}[l_0^2(2ij)+\tfrac{1}{4}\pi^2]^{-1}\left(\ln\left|\frac{2j+(2n+1)}{2j-(2n+1)}\right|\right)^2 + 2(2n+1)\{\ln[(\ln 2\xi p_F)^2+\tfrac{1}{4}\pi^2] - \ln[(\ln(2n+1))^2+\tfrac{1}{4}\pi^2]\}\right.$$

$$-\frac{\pi\xi p_F}{\ln 2\xi p_F}\left(2-\frac{0.88}{\ln 2\xi p_F}\right) + \frac{2n+1}{(\ln 2\xi p_F)^2}\left[(2\ln 2\xi p_F)\ln\left(\frac{2\epsilon_F}{\pi T(2n+1)}\right)+\frac{\pi^2}{24}\right]$$

$$-\frac{\pi T}{2\epsilon_F}(2n+1)[\mathrm{Re}E_1(-\ln 4\xi p_F+\tfrac{1}{2}\pi i) - \mathrm{Re}E_1(-\ln 2\xi p_F+\tfrac{1}{2}\pi i)]$$

$$\left. + \pi\{\mathrm{Re}E_1(-\ln 2\xi p_F+\tfrac{1}{2}\pi i) - \mathrm{Re}E_1[-\ln(2n+1)+\tfrac{1}{2}\pi i]\right\}.$$

After some labor it becomes quite clear that this represents only a slight increase of T_c, which can be estimated to be less than 1% of the depression of the critical temperature due to impurities; furthermore, this small variation is not accessible to experimental confirmation, and therefore there seems to be no point in carrying out more precise numerical computation.

B. Large-Distance Limit

For the behavior of $\Delta(r)$ at large distances from the impurity ($r \gg \xi_0$), a different approach can be used, based on the approximations involved in the derivation of the Ginzburg-Landau equations from the microscopic theory.[16] As pointed out in H this is equivalent to the replacement of the operator $L(r)$ by the following differential operator:

$$L(r) + z \cong -\tfrac{7}{12}\,\zeta(3)\,\nabla_r^2, \quad (24)$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

is the Riemann ζ function. After some algebraic labor, which follows the scheme outlined in H, we obtain

$$f_1(\rho) \cong \frac{24}{7\zeta(3)\pi}\,\frac{\gamma}{(p_F\xi)^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1+\gamma)^2+d^2} \times\left(\frac{1}{\rho}\,\frac{1}{2n+1}\,(e^{-(2n+1)\rho}-1) - E_1[(2n+1)\rho]\right), \quad (25)$$

and all the quantities that appear in the above equation are defined after (15). Since $\gamma,\ d \gg 1$, we can approximate the preceding relation by

$$f_1(\rho) \cong -\frac{24}{7\zeta(3)\pi}\,\frac{1}{(p_F\xi)^2} \times\left[\frac{C_0}{\rho} - \frac{\gamma}{\gamma^2+d^2}\left(\frac{e^{-\rho}}{\rho} - E_1(\rho)\right)\right]. \quad (26)$$

The constant C_0 can be evaluated directly and turns out to be

$$C_0 = \frac{\pi T_c}{2\Gamma}\left(\psi(\lambda_c) + \gamma_E - \frac{\Gamma}{2\pi T_c}\,\zeta(2,\lambda_c)\right), \quad (27)$$

where all used quantities were defined after Eq. (22). This way we see that the behavior of $f_1(\rho)$, as ρ becomes very large, is of the ρ^{-1} type, as expected from considerations based on the Landau-Ginzburg equation for the order parameter.[17] For this reason, and because the validity of the approximation invoked in (24) becomes better and better as ρ grows, it is clear that the asymptotic form above is more reliable than the one given in (16). It also becomes clear that resonance scattering in superconductors produces a spatial variation of the order parameter, near an impurity, of the same type as when the impurity does sustain a magnetic moment. The magnitude of the perturbation though is much smaller in the resonance scattering case, since the comparable nondimensional quantity $(a/\xi)^2$, which governs magnetic scattering (with a being the reduced scattering length), is

about 1000 times bigger than $1/(p_F^2\xi^2\gamma)$, the equivalent magnitude in our problem.

IV. DIFFERENT APPROACH

Because of the very involved nature of the method of solution used in Sec. III, an alternative way of handling the problem seems desirable; the one we present hereafter largely follows the lines suggested by Tsusuki and Tsuneto (TT), but avoids many of its drawbacks, which were discussed and severely criticized by Heinrichs.[1] The basic difference with the preceding calculations is that now we work everything out in Fourier transformed rather than in coordinate space.

We start by taking a three-dimensional Fourier transform of Eq. (12), after combining it with (11), which yields the following expressions:

$$L(k)f(k) = -\frac{4}{m^2 v_F}\frac{\Gamma}{T}\sum_{n=0}^{\infty}\frac{1}{(\omega_n+\Gamma)^2+E_d^2} \times \frac{1}{\xi k}\tan^{-1}\left(\frac{\xi k}{2n+1}\right)\Xi(\omega_n;f), \tag{28}$$

$$\Xi(\omega_n;f) = \frac{T}{\omega_n} + \int \frac{d^3k'}{(2\pi)^3}\frac{1}{\xi k'}\tan^{-1}\left(\frac{\xi k'}{2n+1}\right) f(k'), \tag{29}$$

$$L(k)+z = \frac{2}{\xi k}\sum_{n=0}^{\infty}\left[\frac{\xi k}{2n+1} - \tan^{-1}\left(\frac{\xi k}{2n+1}\right)\right], \tag{30}$$

where $\xi = v_F/2\pi T$, was already defined before.

We now set out to find the asymptotic behavior of $f(k)$ for small values of k compared to ξ^{-1}. Expanding (28) to second order, we obtain

$$L(t)f(t) \cong \frac{4\gamma t^2}{3m^2 v_F \pi}\sum_{n=0}^{\infty}\frac{\Xi(n;f)}{(2n+1)^2[(2n+1+\gamma)^2+d^2]}, \tag{31}$$

where we have introduced the nondimensional wave number $t = \xi k$. In the derivation of the above relation we have used Eq. (13) and the fact that for a single impurity $z = \ln(T_{c0}/T_c)$ is equal to zero, since the volume of the system is of macroscopic size. Furthermore, $L(t)$ for $t \ll 1$, is given by

$$L(t) \cong \tfrac{7}{12}\xi(3)\, t^2, \tag{32}$$

and therefore

$$f(t) \cong \frac{16\gamma}{7\xi(3)\pi m^2 v_F}\sum_{n=0}^{\infty}\frac{\Xi(n;f)}{(2n+1)^2[(2n+1+\gamma)^2+d^2]}, \tag{33}$$

which is independent of t.

For large values of t the summation over n is carried out using the Euler-Maclaurin formula.[18] The approximate expression that is obtained reads

$$L(t)f(t) \sim (\tan^{-1} t)/t\,. \tag{34}$$

Furthermore, in H it was shown that

$$L(t)+z \cong \frac{1}{it}\ln\frac{\Gamma(\frac{1}{2}+\frac{1}{2}it)}{\Gamma(\frac{1}{2}-\frac{1}{2}it)} - \psi(\tfrac{1}{2}), \tag{35}$$

where $\Gamma(v)$ is the well-known γ function. Taking the limit of large t ($t \gg 1$) and using the following asymptotic form for $\ln\Gamma(v)$,[19]

$$\ln\Gamma(v) = (v-\tfrac{1}{2})\ln v - v + \tfrac{1}{2}\ln(2\pi) + \cdots, \tag{36}$$

we obtain

$$L(t) \underset{t\to\infty}{\longrightarrow} \ln(1+t^2), \tag{37}$$

and therefore for $t \gg 1$

$$f(t) \sim -\tan^{-1} t/[t\ln(1+t^2)]\,. \tag{38}$$

The preceding calculations imply that we may choose $f(t)$, for any value of t, as

$$f(t) = -[Ct/(1+t^2)]\,[\tan^{-1} t/\ln(1+t^2)], \tag{39}$$

where C is a constant to be determined self-consistently. This is achieved using (33) in the limit of $t = 0$; we obtain

$$-C = \frac{16\gamma}{7\zeta(3)\pi m^2 v_F}\sum_{n=0}^{\infty}\frac{\Xi(n;f)}{(2n+1)^2[(2n+1+\gamma)^2+d^2]}, \tag{40}$$

and substituting the expression for Ξ given in (29) we find

$$C \cong \frac{2}{\pi^2 m^2 v_F}\frac{\gamma}{\gamma^2+d^2}\,[I(t_c)-1]^{-1}\,. \tag{41}$$

We have used the fact, already mentioned before, that γ, $d \gg 1$, and we have defined

$$I(t_c) = \frac{8}{7\zeta(3)m^2\pi^2 v_F}\frac{\gamma}{\gamma^2+d^2}\int_0^{t_c}\frac{dt\,t^2}{1+t^2}\frac{\tan^{-1} t}{\ln(1+t^2)} \times \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\tan^{-1}\left(\frac{t}{2n+1}\right). \tag{42}$$

Here a most serious drawback of this method of calculation appears, because we had to introduce a cutoff in wave-number space in order to avoid the mild [$\int^{\infty}(\ln x)^{-1}dx$ type] divergence at infinity. This is equivalent to excluding short distances in real space, where we know that all our expressions, starting from Eq. (5) on, are only approximately valid; we attribute the divergence of $I(t_c)$ to these approximations.

The Fourier transform $f(\rho)$ of $f(t)$, we are looking for, is then

$$f(\rho) = -C\frac{4\pi}{\xi^3}\frac{1}{\rho}\int_0^{\infty}\frac{dt\,t^2}{1+t^2}\frac{\tan^{-1} t}{\ln(1+t^2)}\sin(t\rho)\,, \tag{43}$$

where again $\xi = v_F/2\pi T$, and C is given by (41). We notice that the integral, which has to be evaluated numerically, for $\rho \gg 1$ is only weakly dependent on ρ so that the ρ^{-1} spatial dependence of the order parameter, at large distance from the impurity, is found again. As mentioned before, we do not expect to obtain the correct form of $f(\rho)$, for small values of ρ, by this method.

V. CONCLUSION

The detailed calculations that have been carried out in the previous sections can be summarized as follows. The superconducting order parameter in alloys of simple metals with nonmagnetic transition-metal impurities, has a long-range spatial variation; this spatial dependence is similar in shape (but much smaller in magnitude) to the case in which the impurity does sustain a magnetic moment in the host matrix. At large distance the behavior is dominantly of the ρ^{-1} type, where ρ is a reduced distance expressed in units of the typical length $\xi = v_F/2\pi T$, which (near the transition temperature) is approximately equal to the superconducting coherence length. This ρ^{-1} dependence is consistent with arguments advanced by Caroli *et al.*[17] on the basis of the Ginsburg-Landau equation.

The long-range variation of $\Delta(r)$ is important to understand in relation to the explanation of the marked initial decrease $\partial T_c/\partial n_I$ of the superconducting critical temperature, in alloys containing nonmagnetic impurities with broadened d levels. The explanation of this marked decrease on the basis of resonance scattering has brought about a long-standing controversy on the validity of the mechanism.

The arguments advanced were based on Anderson's theorem[9] which is usually stated as follows: "If a static external perturbation does not break the time-reversal invariance symmetry and does not cause a long-range spatial variation of the order parameter, the thermodynamic properties of the superconductor remain unchanged in the presence of the perturbation."[10] To the best of our knowledge, the spatial variation of the order parameter in relation to resonance scattering had not been investigated before; neither had the consequences of this spatial variation for the decrease in critical temperature been examined. Our work leads us to the conclusion that the change in superconductor thermodynamics, due to spatial variation of the order parameter, is very small. The main effect caused by impurities with broad d levels is due to the incoherent d-state admixture, which strongly enhances the total density of states over the Fermi surface. Therefore, care should be taken when using the Anderson theorem since it is only valid if the total density of states at the Fermi level has no incoherent admixture in the superconducting state.

The solution schemes of Heinrichs[2] and a slightly modified version of the one proposed by Tsusuki and Tsuneto[1] have been used. The first one is quite rigorous and leads to physically correct answers which, however, are quite involved algebraically and therefore difficult to handle and use. The method of TT is more simple, predicts physically reasonable answers only at long distances from the impurity, and requires the introduction of a cutoff in configuration space, excluding a region near the impurity, due to the approximations that are used to make the problem a tractable one.

Several extensions are possible and work on them is at present in progress. This is due to the fact that we have not taken into account Coulomb correlations in the superconducting state[5,6] nor localized spin fluctuations (LSF) on the impurity which are known to be important for alloys like *Al*Mn.

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Current-Induced Flow of Domains in the Intermediate State of a Superconductor

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Current-induced flow of superconducting domains similar to the flow of normal domains (flux flow) has been observed in the intermediate state. A theory (neglecting the Hall effect) is presented for the motion of domains in an intermediate state of arbitrary topology. Without pinning, the current in the normal regions is uniform and equal to J_0, the average current density. Domains move with velocity $v_D = cJ_0/\sigma H_c$. Both results agree with those previously derived for flux flow. Introducing pinning gives agreement between the predicted and measured velocities.

A dynamic current-carrying intermediate state was first discussed by Gorter,[1] who suggested that a transport current could induce motion of laminae perpendicular to the current. Although attempts to observe this particular phenomenon have produced conflicting results,[2-5] current-induced motion of simply connected normal domains (flux flow) has been demonstrated by several experiments.[5,6] Recently, we have observed that a transport current can also induce a flow of simply connected superconducting domains,[7] and it appears from our observations that any intermediate-state topology is unstable with respect to transverse motion in the presence of a transport current. In this paper we describe the characteristics of superconducting domain flow and present a theory of the current-induced motion of domains of arbitrary topology. In the theory, the magnetic field is assumed to be perpendicular to the transport current and the Hall effect has been neglected. The electric field within the normal regions is calculated using Maxwell's equations and the condition that the electric and magnetic fields vanish within the superconducting regions. Using this electric field, the domain velocity $\vec{v}_D$ is obtained by solving a power-balance equation which equates the power supplied by a battery with the power dissipated by Joule heating and by pinning or motion-induced thermal gradients.[8] This formulation avoids the difficult problem of defining the force on a domain.[9] For normal domains, where there is no dissipation in the absence of motion, the force has been obtained by thermodynamic arguments,[10] but this approach is not directly applicable to superconducting domains where there is dissipation in the absence of motion.

The intermediate state was observed using the magneto-optic rotation in a thin film of $EuSe_{(0.9)}$ $EuF_{2(0.1)}$ evaporated onto the sample surface.[11]

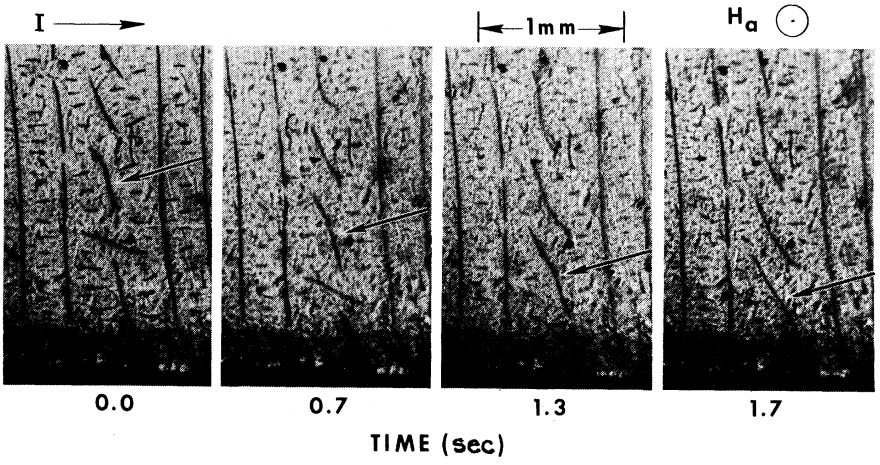

FIG. 1. Four sequential photographs illustrating the motion of superconducting domains (dark regions). The arrows point to the same domain in each picture. The sample is a rectangular Pb slab 4×12×40 mm. The small superconducting inclusions in the left-hand picture have been retouched for illustrative purposes.